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Travelling waves and dynamic scaling in a singular interface equation: analytic results

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Abstract. Analytic results are obtained for Zhang's singular interface equation $u_t = u_{xx} + \ln|u_x|$. We examine in detail travelling wave solutions $u(x - \lambda t)$ and self-similar solutions of the form $u = tg(\frac{x}{\sqrt{t}}) + f(t)$. There are solutions with $u_x = 0$ at some points where u_{xx} blows up. For general initial data these solutions seem to play the role of intermediate asymptotics.

1. Introduction

The roughening of initially smooth interfaces has recently attracted great interest. As was shown by Kardar, Parisi and Zhang a relatively simple nonlinear partial differential equation with a stochastic term could successfully be used to describe the major features of the process of roughening in a wide class of surface growth phenomena [1]. Under some conditions, however, the apparently stochastic development of the interface is not due to an external noise, but is a result of an underlying instability. Thus, a possible alternative description of rough surface growth can be based on simple *deterministic* partial differential equations containing a singular or unstable term. A well known example of this type is the Kuramoto–Sivashinsky equation [2] where an unstable Laplacian term is balanced by a stabilizing term of a higher order, $-\Delta^4 u$, resulting in a complicated spatio-temporal behaviour [3, 4]. Here we shall consider the singular interface equation of Zhang [5], which was introduced in a study on complex directed polymers:

$$u_t = u_{xx} + \ln|u_x| \tag{1}$$

and a closely related more general equation

$$u_t = u_{xx} + \delta u_x^2 + \ln|u_x| \qquad \delta > 0. \tag{2}$$

These equations are interesting because they can be considered as perhaps the simplest examples of self-sustained complex behaviour without a noise term. The *discretized* version of equation (1) has been shown to exhibit an interesting scaling behaviour [5].

The finite difference version of a slightly modified form of equation (1) has been numerically investigated for the Cauchy–Dirichlet problem with periodic boundary conditions. Here the initial function was a nonnegative noise function with $u_x(x, 0) = 0$ at several points [6]. These calculations (besides other results) have shown that after some time (large t) the typical form of the solution is like the one in figure 1. In the neighbourhood of the two local minima the solution looks like a linear function which seems to be logical because x and $-x$ are solutions of equation (1). On the other hand, the function $|x|$ is not

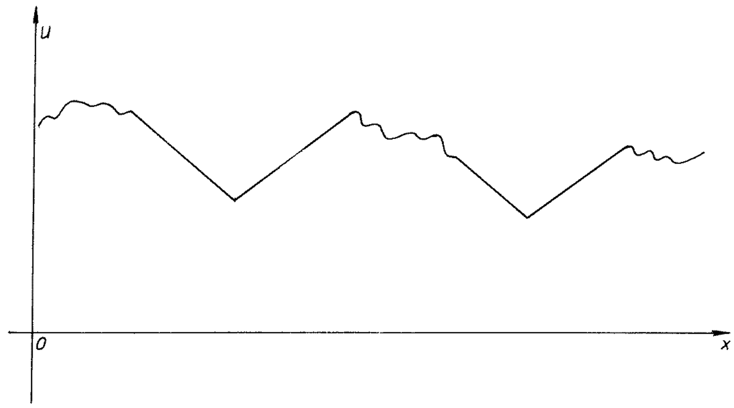

Figure 1. Typical form of the numerical solution to equation (1) for some fixed (large) t which often occurs and remains unchanged around the two minima. Since this is a sematic figure, there is no scale on the axes. In a typical simulation the value of x can vary between zero and a few hundred, while the difference between the maximal and minimal value of the solution can change in a wide range from 0.01 to 1 depending on the parameters (see [6]).

a classical solution due to a lack of smoothness at zero ('the flux is not continuous'). It is easy to see that $|x|$ is not a weak solution either (see the definition in the next section).

Our results show the existence of special solutions, namely travelling waves (TW) and self-similar solutions which have continuous first derivatives at singular points, consequently they are solutions in the weak sense. Some of our special solutions have linear asymptotics, therefore they look like the numerical solution (in figure 1) in the neighbourhood of the two minima. The natural conclusion is that these solutions are good candidates for being intermediate asymptotics [7].

In the next section we give a complete picture of TW solutions to equations (1) and (2). Basically there are two types of such solutions: for the first one u_x never becomes zero, while for the second one u_x will be zero at some points. The second-type solutions have linear asymptotics for large x and fixed t and at the singular points where $u_x = 0$ their first derivatives are continuous (unlike the function $|x|$). However, they do not have bounded second derivatives (u_{xx} blows up at the singular points).

In the final section we present an 'almost explicit' self-similar-type solution to equation (1). At the points where $u_x = 0$ it behaves exactly like the corresponding TW solutions but for large x (t fixed) it is like x^2.

2. Travelling waves

We shall see that $u(x,t)$ has no continuous second derivative with respect to x at the points where $u_x = 0$. In the neighbourhood of such points we indicate in what sense the solutions $u(x,t)$ satisfy equations (1) and (2). This is the usual definition of weak solutions in the theory of nonlinear partial differential equations [8]. Identity (3) is the result of multiplying (2) by $\varphi(x,t)$ and the formal integration by parts.

Definition. We shall say that $u(x,t)$ is a *weak solution* to equation (2) if the integral identity

$$\int_{x_0}^{x_1} u\varphi\Big|_{t_0}^{t_1} \mathrm{d}x = \int_{x_0}^{x_1}\int_{t_0}^{t_1} (u\varphi_t - u_x\varphi_x + \delta u_x^2\varphi + \varphi \ln|u_x|)\, \mathrm{d}x\, \mathrm{d}t \tag{3}$$

is satisfied for all rectangles $R = [x_0, x_1] \times [t_0, t_1] \in (-\infty, \infty) \times (0, \infty)$ and smooth (on R) functions $\varphi(x, t)$ such that $\varphi(x_0, t) = \varphi(x_1, t) = 0$.

Note that this definition requires only the integrability of functions u, u_x, u_x^2, $\ln|u_x|$ and does not contain u_{xx}.

The TW solution to equations (1) and (2) is a solution of the form $u = g(\xi)$, where $\xi = x - \lambda t$ and λ is a real number (speed).

The function $g(\xi)$ satisfies the following second-order ordinary differential equation

$$g'' + \lambda g' + \delta g'^2 + \ln|g'| = 0. \tag{4}$$

Using the substitution $g' = f$ we get

$$f' + \lambda f + \delta f^2 + \ln|f| = 0. \tag{5}$$

First, we consider the case $\delta = 0$ (equation (1)) which turned out to be more general from our point of view. Then equation (5) reads

$$f' = -\lambda f + \ln\frac{1}{|f|}. \tag{6}$$

Here we shall study equation (6) with $\lambda > 0$ in detail; the $\lambda < 0$ case can be handled analogously.

The roots of the nonlinear equation

$$-\lambda f + \ln\frac{1}{|f|} = 0 \tag{7}$$

determine the equilibrium points (where $f' = 0$) of (6) .

Remark 1. If the constants f_i are equilibrium points of (6) then the linear functions $u = f_i(x - \lambda t)$ will be TW solutions to equation (1).

In the following we shall consider the nontrivial TW solutions to (1).

Remark 2. In each interval not containing any of the equilibrium points, the function

$$F(f) := \int^f \frac{\mathrm{d}s}{\ln\frac{1}{|s|} - \lambda s} = \xi \tag{8}$$

is strictly monotone (increasing or decreasing). So the inverse function, F^{-1}, exists in this interval and the solution to equation (6) is the function

$$f(\xi) = F^{-1}(\xi)$$

which gives us the solution to (1):

$$u = g(\xi) = \int f(\xi)\,\mathrm{d}\xi + c \qquad c \text{ is a constant.}$$

Let $\lambda_0 = e^{-1}$, ($\ln e = 1$). Depending on the speed of the travelling wave λ, we shall have three different cases.

I. If $0 < \lambda < \lambda_0$ then equation (6) has three equilibrium points. Namely equation (7) has exactly three roots: f_0, f_1, f_2, where $0 < f_0 < 1$, $f_1 < -1$ and $f_2 < f_1$.

II. If $\lambda = \lambda_0$ then equation (7) has two roots: f_0, f_1, where $0 < f_0 < 1$, $f_1 = -e$.

III. If $\lambda > \lambda_0$ then we have only one equilibrium point $0 < f_0 < 1$.

Case I: $0 < l\lambda < \lambda_0$. Here equation (6) has four different solutions separated by equilibrium solutions f_i $(i = 0, 1, 2)$. The solution between f_1 and f_0, is the only one which becomes zero at one point. Because $f = g' = u_x$, the original equation (1) is genuinely singular there.

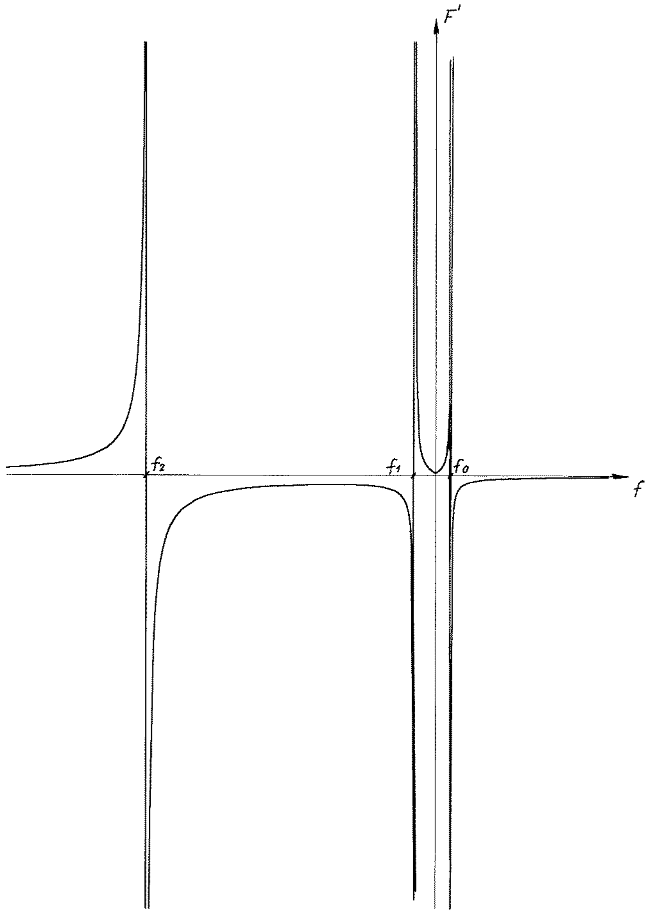

Figure 2. Function $F'(f)$ used to construct $u(x - \lambda t)$, the TW solutions of equation (1) in the case where $0 < \lambda < \lambda_0$.

In the following we construct TW solutions by using function $F(f)$ of (8) as described in remark 2. The starting point of our proof is the function $F'(f)$ (see figure 2).

(i) $-\infty < f < f_2$. In this interval $F' > 0$, $F' \to +\infty$ when $f \nearrow f_2$ and $F' \searrow 0$ when $f \to -\infty$. The function, F, is strictly monotone increasing, $F \to +\infty$ if $f \nearrow f_2$ as $-\ln|f - f_2|$ and $F \to -\infty$ for $f \to -\infty$ as $-\ln(-f)$. Consequently, the function $f(\xi)$ for large $\xi > 0$ behaves like $-e^{-\xi} + f_2$, for large negative ξ such as $-e^{-\xi}$. Integration gives the function $u = g_1(\xi)$ which has asymptote $f_2\xi$ if $\xi \to +\infty$ and is like $e^{-\xi}$ for large negative ξ.

(ii) $f_0 < f < +\infty$. Here $F' < 0$, F is strictly decreasing, $F \sim -\ln|f - f_0|$ if $f \searrow f_0$, and F behaves like $-\ln f$ for large f.

Hereafter $a(x) \sim b(x)$ means that $\lim_{x\to 0} \frac{a(x)}{b(x)} = c$, where c is a positive constant.

The function $f(\xi)$ decreases from ∞ to f_0 and globally behaves like $e^{-\xi} + f_0$. The corresponding solution, $u = g_2(\xi)$, has $f_0\xi$ as its asymptote for large positive ξ and behaves like $-e^{-\xi}$ for large negative ξ. Thus, basically, g_2 looks like $-g_1(\xi)$.

(iii) $f_2 < f < f_1 < -1$. In this case, as it is easy to see, we have a bounded negative $f(\xi)$ such that $f(-\infty) = f_1$ and $f(+\infty) = f_2$. Thus, $u = g_3(\xi)$ is like a hyperbole with

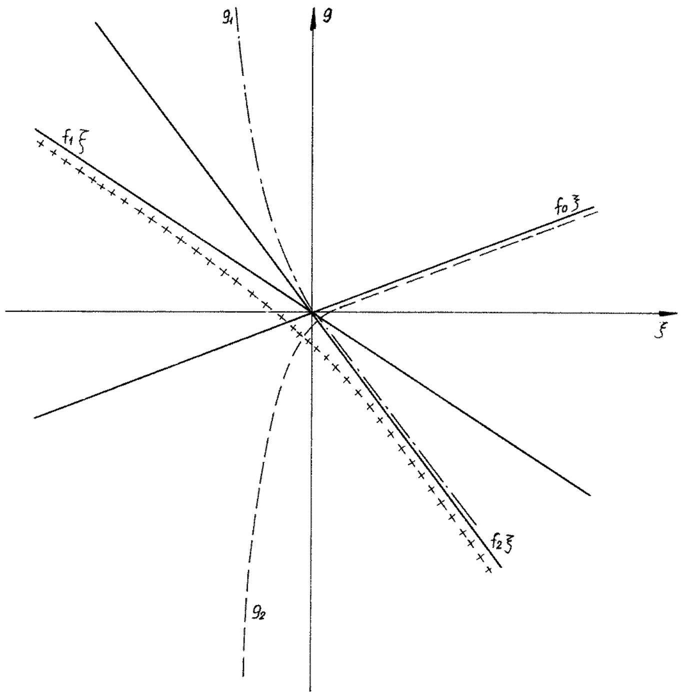

Figure 3. Travelling wavesolutions ($u = g_1(\xi)$ is denoted by — · —, $u = g_2(\xi)$ is – – –, while $u = g_3(\xi)$ is + + +, $\xi = x - \lambda t$) of equation (1) for the case when u_x never turns to zero.

asymptotes $f_1\xi$ for large negative ξ and $f_2\xi$ for $\xi \to +\infty$.

The TW solutions, $g_1(\xi)$, $g_2(\xi)$, $g_3(\xi)$ for which u_x never turns to zero, are presented in figure 3.

(iv) $f_1 < f < f_0$. This is the first important case: the derivative of the corresponding solution becomes zero at some point (without a loss of generality, we can suppose that it happens at the origin $x = 0$).

In the interval (f_1, f_0) the function $F'(f)$ is nonnegative, $F'(0) = 0$ and $F''(\pm 0) = \pm\infty$. In the neighbourhood of f_0 and f_1 the function F' behaves like $|f - f_i|^{-1}$. The function F is strictly monotone increasing, $F \to -\infty$ if $f \to f_1$ and $F \to +\infty$ for $f \to f_0$. Consequently, the function $u = g_4(\xi)$ decreases to zero and is increasing from zero. It is a nonnegative hyperbole-like function with asymptotes $f_1\xi - \xi_0$ from the left and $f_0\xi - \xi_1$ from right, $\xi_i > 0$, see figure 4.

The function $F'(f)$ for small f behaves like $(\ln \frac{1}{|f|})^{-1}$, thus $F(f) \sim f(\ln \frac{1}{|f|})^{-1}$. It is easy to see that $f(\xi) \sim |\xi| \ln \frac{1}{|\xi|}$ for small $|\xi|$, so

$$g_4(\xi) \sim \xi^2 \ln \frac{1}{|\xi|} \qquad \text{for small } \xi.$$

One can see that the second derivative of $g_4(\xi)$ at zero is not continuous: $g_4''(\xi) \sim \ln \frac{1}{|\xi|} \to +\infty$ when $|\xi| \to 0$. In contrast to (1), identity (3) does not contain u_{xx} and all the integrals in (3) exist except, possibly, the last one $\int \varphi \ln |u_x| \, dx$. If $\varphi = 1$ in the ε-neighbourhood of zero (we suppose that $u_x = 0$ at zero), then the convergence of this integral is equivalent

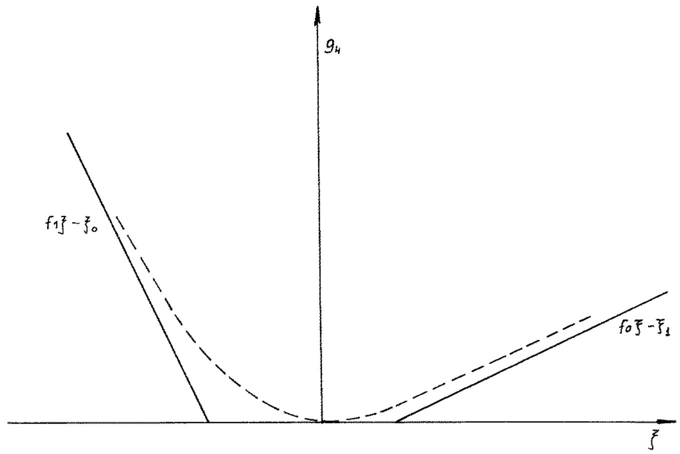

Figure 4. Travelling wave solution of equation (1) when $u_x = 0$ at some point (without loss of generality we suppose that $u = 0$ at $x = 0$, for fixed t). It has the cusp-like form similar to the ones obtained by numerical solutions (compare with figure 1).

to the convergence of $\int_{-\varepsilon}^{\varepsilon} \ln\left(|x| \ln \frac{1}{|x|}\right) \mathrm{d}x$. However,

$$\left|\ln\left(|x|\ln\frac{1}{|x|}\right)\right| \leqslant |\ln|x|| + \left|\ln\ln\frac{1}{|x|}\right| \leqslant c|\ln|x||$$

which is integrable at zero.

Case II $\lambda = \lambda_0$. Equation (7) has two zeros: $f_0 \in (0, 1)$ and $f_1 = -e$. The function $F'(f)$ is the same as in figure 2 with $f_2 = f_1$. Thus, we have three travelling waves which behave qualitatively like g_1, g_2 and g_4 from the previous case.

Case III $\lambda > \lambda_0$. The travelling wave corresponding to $f > f_0$ is like g_2. Let us now assume that $f < f_0$. The function $F'(f)$ in the interval $[-\varepsilon, f_0)$, $\varepsilon > 0$ behaves like $F'(f)$ in figure 2, but decreases to zero when $f \to -\infty$ like $-\frac{1}{f}$ having a maximum at some point $f_3 < -\varepsilon$. The corresponding solution, $u = g_5(\xi)$, behaves like g_4 in the neighbourhood of zero, while for large positive ξ it has a linear asymptote $f_0\xi - \xi_0$ and for large negative ξ behaves like $e^{-\xi}$.

In the case $\lambda = 0$ (stationary TW) we have $F'(f) = -\frac{1}{\ln|f|}$. Here there are two equilibria $f_0 = 1$, $f_1 = -1$. The travelling wave between $f_1 < f < f_0$ is qualitatively the same as g_4 while for $f_0 < f$ the solution is like g_2.

When $f < -1$ the TW solution $u = g_6(\xi)$ is a strictly monotone decreasing function having ξ as an asymptote for large negative ξ and behaves like $\xi^2 \ln \frac{1}{\xi}$ for large positive ξ.

The case $\delta > 0$ (equation (2)), $\lambda > 0$ is qualitatively the same as the previous one ($\delta = \lambda = 0$): one has two equilibria, f_3 and f_4, such that $f_4 < 0 < f_3 < 1$.

Remark 3. In order to make the effect caused by the ln term in equation (1) more transparent, it is worth comparing the TW solutions of (1) with the TW solutions of the heat equation. Equation $u_t = u_{xx}$ has two independent positive TW solutions e^{x+t} and $e^{-(x-t)}$. Consider the first one: It grows (decays) exponentially when $x \to +\infty$ ($x \to -\infty$). The term $\ln|u_x|$ is a source term for $|u_x| > 1$ and plays the role of an absorber for $|u_x| < 1$ and changes the behaviour of the solution in a nonsymmetric way. For instance, let us take the TW solution $g_4(\xi)$ from case I where $f_0 \in (0, 1)$ and $f_1 < -1$. For large negative ξ one has

$|u_x| \sim |f_1| > 1$ consequently $\ln|u_x|$ is a source and instead of exponential decay one gets linear growth: $u \sim f_1 x$. For large positive x we have $|u_x| \sim |f_0| < 1$ and the exponential growth is replaced by the linear one.

3. Self-similar-type solutions

This section deals with a self-similar-type solution to equation (1) and describes its geometrical properties. This solution will have the form

$$u(x,t) = (t+t_0)^{\alpha} g\left(\frac{x}{(t+t_0)^{\beta}}\right) + f(t) \tag{9}$$

where the functions $g(\xi)$, $f(t)$ and the positive constants α, β are to be determined.

Formula (9) expresses the *self-similar* nature of the solution, or can be regarded as *solution with dynamic scaling*. Here we refer to dynamic scaling because equation (9) has a form which is analogous to the scaling behaviour of self-affine growing surfaces (see, e.g. [9, equation (7.19)]) whose properties can be described in terms of dynamic scaling.

Theorem. Let $g(\xi)$ be the solution of the generalized Weber equation

$$g(\xi) - \tfrac{1}{2}\xi g'(\xi) = g''(\xi) + \ln|g'(\xi)|. \tag{10}$$

Then

(1) the function

$$u(x,t) = (t+t_0)[g(\xi) + \tfrac{1}{2}(\ln(t+t_0) - 1)] \qquad \xi = \frac{x}{\sqrt{(t+t_0)}}$$

satisfies equation

$$u_t = u_{xx} + \ln|u_x|.$$

(2) Function $g(\xi)$ has the following geometrical properties

(i) $g(\xi)$ is monotonously increasing for $\xi > 0$ and $g(\xi)$ is monotonously decreasing for $\xi < 0$,

(ii) $g(\xi) \sim \xi^2 \ln\frac{1}{|\xi|}$, for small $|\xi|$,

(iii) $g(\xi) \sim \xi^2$, for large $|\xi|$.

Remark 4. We see that $g(\xi)$ is roughly ξ^2 (we suppose, as before, that $g(0) = 0$). The only important difference is the nonsmoothness of g at zero: the second derivative blows up at $\xi = 0$. The next to leading-order term in (ii) is $|\xi^3|\ln\frac{1}{|\xi|}$.

Remark 5. It follows from (10) that if $g(\xi)$ is a solution then $h(\xi) = g(-\xi)$ also satisfies (10). Since $g \equiv$ constant is not a solution, it is possible to show, that for $\xi > 0$ the initial value problem for (10) with $g(0) = 0$, $g'(0) = 0$ has a unique global solution $g = g^+(\xi)$. For $\xi < 0$ we set $g = g^+(-\xi)$. At the point $\xi = 0$ the function $g(\xi)$ is not necessarily smooth ($g''(0)$ can blow up) but we can understand equality (10) in the neighbourhood of $\xi = 0$ in a weak sense, as in the definition of the weak solution for (1).

Proof of equation (1). Substitution into (1) gives

$$\begin{aligned}\alpha(t+t_0)^{\alpha-1}g(\xi) - \beta(t+t_0)^{\alpha-1}\xi g'(\xi) + f'(t) &= (t+t_0)^{\alpha-2\beta}g'' \\ +\ln|(t+t_0)^{\alpha-\beta}g'(\xi)| &= (t+t_0)^{\alpha-2\beta}g'' + (\alpha-\beta)\ln|(t+t_0)| + \ln|g'(\xi)|.\end{aligned}$$

Now, one can see that if we take

$$f(t) = \tfrac{1}{2}(t+t_0)[\ln(t+t_0) - 1] \qquad t_0 \geqslant e,\ \alpha = 1 \text{ and } \beta = \tfrac{1}{2}$$

$u(x,t)$ will be a solution to (1) provided that $g(\xi)$ satisfies equation (10). □

Proof of equation (2).

(i) It is sufficient to show that $g'(\xi) > 0$ for $\xi > 0$.

In fact, suppose that ξ_1 is the first value where $g'(\xi)$ is zero. Let us take $\xi_2 = \xi_1 - \varepsilon > 0$, where ε is a small positive number. One has $g''(\xi_2) < 0$ and $\ln|g'(\xi)|$ is a large negative number. However, in the same neighbourhood the left-hand side of (10) is positive. So $g'(\xi) > 0$ for $\xi > 0$.

(ii) We can see from (10) that the behaviour of $g(\xi)$ in the right neigbourhood of $\xi = 0$ ($\xi > 0$) is controlled by the equation

$$g_1''(\xi) + \ln|g_1'(\xi)| = 0.$$

By setting $g_1'(\xi) = y(\xi)$, one has

$$y'(\xi) + \ln|y(\xi)| = 0$$

from which we have $y(\xi) \sim \xi \ln \frac{1}{\xi}$, for $\xi \in (0, \varepsilon)$, $\varepsilon > 0$ is small, so

$$g_1(\xi) \sim g(\xi) \sim \xi^2 \ln \frac{1}{\xi} \qquad \text{for } \xi \in (0, \varepsilon).$$

Thus, $g(\xi)$ at $\xi = 0$ (where $g' = 0$) behaves like $\xi^2 \ln \frac{1}{|\xi|}$, which means that $u(x, t)$ is like $x^2 \ln \frac{1}{|x|}$ for fixed t at $x = 0$ (where $u_x = 0$). This proves (ii).

(iii) First we show that $g(\xi)$ is at least power-like at infinity. Suppose the contrary:

$$\frac{g(\xi)}{\xi^\varepsilon} \to 0 \qquad \text{for all } \varepsilon > 0 \tag{11}$$

i.e. that $g(\xi)$ grows slower than any power of ξ. In that case, because of the monotonicity of g, we have $g'(\xi) \to 0$ as $\xi \to \infty$. Indeed, if $g'(\xi) \geqslant c > 0$ for $\xi > \xi_0$ then integration gives $g \geqslant c\xi$ which contradicts (11). But if $g'(\xi) \to 0$, then the right-hand side of (10) goes to minus infinity. The only term of (10) which is able to balance it, is $-\frac{1}{2}\ \xi\ g'$, consequently $g'(\xi) \sim \frac{2}{\xi} \ln \xi$ for large ξ and $g \sim (\ln \xi)^2$. Substituting this into (10) and taking ξ large enough we obtain a contradiction.

Substitution $g(\xi) = \xi^\alpha$ into equation (10) gives

$$\xi^\alpha - \frac{\alpha}{2}\xi^\alpha = \alpha(\alpha - 1)\xi^{\alpha-2} + \ln|\alpha\xi^{\alpha-1}|$$

which is true in the limit $\xi \to \infty$ provided that $\alpha = 2$. For a more exact result, suppose that g has the form $g = \xi^2 h(\xi)$ where $\lim_{\xi\to\infty} \frac{h(\xi)}{\xi^\varepsilon} = 0$ for all $\varepsilon > 0$. Substitution into (10) leads to the equation

$$-\frac{1}{2}\xi h' = h'' + \frac{2h}{\xi^2} + \frac{4h'}{\xi} + \frac{1}{\xi^2} \ln|2\xi h + \xi^2 h'|$$

from which we can see that the behaviour of $h(\xi)$ at infinity is controlled by the equation

$$-\tfrac{1}{2}\xi h' = h''$$

having the explicit solution $h(\xi) = \int_0^\xi \mathrm{e}^{-\tau^2/4}\,\mathrm{d}\tau$ with $h(+\infty) = \text{constant}$. $\square$

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